Subexponential parameterized algorithms

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Abstract

We give a review of a series of techniques and results on the design of subexponential parameterized algorithms for graph problems. The design of such algorithms usually consists of two main steps: first find a branch- (or tree-) decomposition of the input graph whose width is bounded by a sublinear function of the parameter and, second, use this decomposition to solve the problem in time that is single exponential to this bound. The main tool for the first step is Bidimensionality Theory. Here we present the potential, but also the boundaries, of this theory. For the second step, we describe recent techniques, associating the analysis of sub-exponential algorithms to combinatorial bounds related to Catalan numbers. As a result, we have $2^{O(\sqrt{k})} \cdot n^{O(1)}$ time algorithms for a wide variety of parameterized problems on graphs, where n is the size of the graph and k is the parameter.

1 Introduction

The theory of fixed-parameter algorithms and parameterized complexity has been thoroughly developed during the last two decades; see e.g. the books of Downey and Fellows [31], Flum and Grohe [35], and Niedermeier [47].

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Usually, parameterizing a problem on graphs is to consider its input as a pair consisting of the graph G itself and a parameter k. Typical examples of such parameters are the size of a vertex cover, the length of a path or the size of a dominating set (the formal definitions of these parameters are detailed in Section 2). Roughly speaking, a parameterized problem on an n-vertex graph with parameter k is *fixed parameter tractable* if there is an algorithm solving the problem in $f(k) \cdot n^{O(1)}$ steps for some function f that depends only on the parameter.

While there is a strong evidence that most fixed-parameter algorithms cannot have running times $2^{o(k)} \cdot n^{O(1)}$ (see [13, 35, 43]), for planar graphs it is possible to design subexponential parameterized algorithms with running times of the type $2^{O(\sqrt{k})} \cdot n^{O(1)}$ (see [13, 15] for further lower bounds on planar graphs). For example, PLANAR k-VERTEX COVER can be solved in $O(2^{3.57\sqrt{k}}) + O(n)$ steps, PLANAR k-DOMINATING SET can be solved in $O(2^{11.98 \cdot \sqrt{k}}) + O(n^3)$ steps, and PLANAR k-LONGEST PATH can be solved in $O(2^{10.52 \cdot \sqrt{k}} \cdot n) + O(n^3)$ steps [27]. Similar algorithms are now known for a wide class of parameterized problems, not only for planar graphs, but also for several other sparse graph classes.

The first paper in this area is due to Alber et al. [2] and it appeared in 2000. Since that work, the study of fast subexponential algorithms attracted a lot of attention. In fact, it not only offered a good ground for the development of parameterized algorithms, but it also prompted combinatorial results, of independent interest, on the structure of several parameters in sparse graph classes such as planar graphs [1, 3, 5, 14, 19, 34, 37, 42, 44] bounded genus graphs [20, 36], graphs excluding some single-crossing graph as a minor [26], apex-minor-free graphs [18] and H-minor-free graphs [17, 20, 21].

We here present general approaches for obtaining subexponential parameterized algorithms (Section 2) and we reveal their relation with combinatorial results related to the Graph Minors project of Robertson and Seymour. All these algorithms exploit the structure of graph classes that exclude some graph as a minor. This was used to develop techniques such as Bidimensionality Theory (Section 3) and the use of Catalan numbers for better bounding the steps of dynamic programming when applied to minor closed graph classes (Sections 4 and 5).

2 Preliminaries

We consider graphs that do not have loops or multiple edges. Given a graph G, we denote its vertex set by $V(G)$ and its edge set by $E(G)$ and we set $n = |V(G)|$. Also for an edge set $F \subseteq E(G)$ we definefe the subgraph of G *induced* by F as the graph whose vertices are the endpoints of the edges in F and whose edges are the edges in F. Given an edge $e = \{x, y\}$ of a graph G, the graph G/e is obtained from G by contracting the edge e, i.e. the endpoints x and y are replaced by a new vertex v_{xy} which is adjacent to the old neighbors of x and y (except from x and y). A graph H obtained by a sequence of edge-contractions is said to be a *contraction* of G. We say that H is a *minor* of G if H is a subgraph of a contraction of G. We use the notation $H \preceq G$ (resp. $H \preceq_c G$) when H is a minor (a contraction) of G. It is well known that $H \preceq G$ or $H \preceq_c G$ implies $\mathbf{bw}(H) \leq \mathbf{bw}(G)$ (for example, in Figure 1, it holds that $G_3 \preceq G_2 \preceq G_1$, $G_2 \preceq_c G_1$ but also $G_3 \not\preceq_c G_2$ and $G_3 \not\preceq_c G_1$).

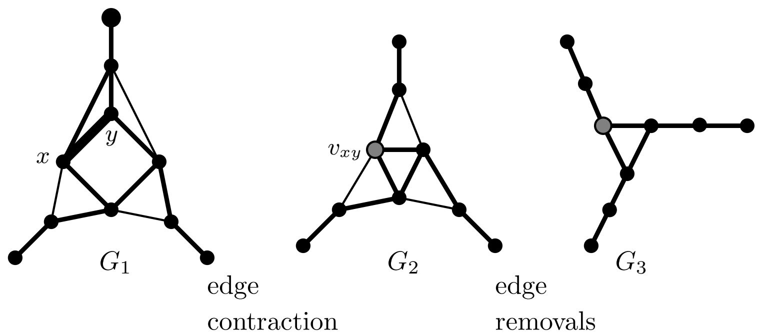

Figure 1: An example of edge contraction and removals.

Given two graphs G and H, the problem of checking whether $H \preceq G$, when both G and H are part of the input is NP-complete (asking for a hamiltonian cycle is equivalent to asking whether G contains as a minor a cycle with n vertices. Similarly, the problem of checking whether $H \preceq_c G$, is NP-complete ([39], problem [GT51]). In the case where H is not a part of the input, checking whether $H \preceq G$ is solvable in $O(n^3)$ where the constants hidden in the "O"-notation heavily depend on the fixed graph H (see Robertson and Seymour [48, 53]). Things are less clear for the problem of asking whether $H \preceq_c G$ when H is not part of the input: there are choices of H where the problem remains NP-complete (see Levin et al. [45]).

We say that a graph G is *H-minor-free* when it does not contain H as a minor. We also say that a graph class $\mathcal{G}$ is *H-minor-free* (or, excludes H as a minor) when all its members are H-minor-free. E.g., by Kuratowski's theorem, the class of planar graphs is a K_5-minor-free graph class and also a $K_{3,3}$-minor-free graph class.

Let G be a graph on n vertices. A *branch decomposition* (T, μ) of a graph G consists of an unrooted ternary tree T (i.e. all internal vertices are of degree three) and a bijection $\mu : L \to E(G)$ from the set L of leaves of T to the edge set of G. We define for every edge e of T the *middle set* $\mathbf{mid}(e) \subseteq V(G)$ as follows: Let T_1 and T_2 be the two connected components of $T \setminus \{e\}$.

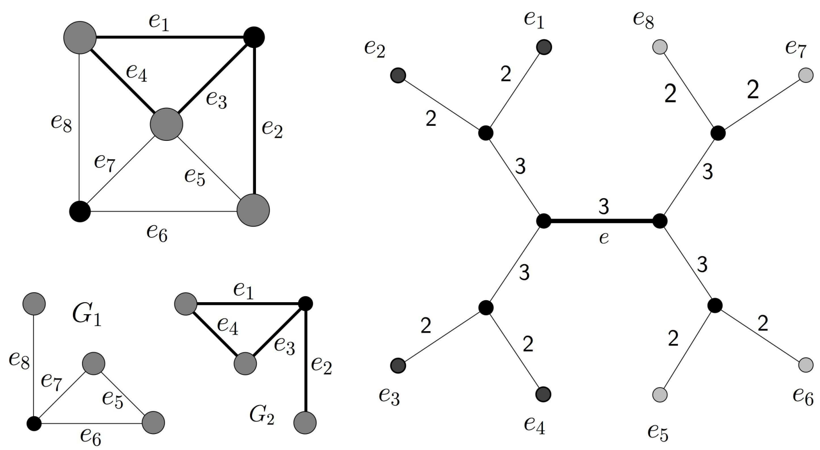

Figure 2: A graph and its branch decomposition of width 3 and the middle set for the edge e.

Then let G_i be the graph induced by the edge set $\{\mu(f) : f \in L \cap V(T_i)\}$ for $i \in \{1, 2\}$. The *middle set* is the intersection of the vertex sets of G_1 and G_2, i.e., $\mathbf{mid}(e) := V(G_1) \cap V(G_2)$. The *width* of (T, μ) is the maximum order of the middle sets over all edges of T, i.e.,

$$\mathbf{w}(T, \mu) := \max\{|\mathbf{mid}(e)| \colon e \in T\}.$$

An optimal branch decomposition of G is defined by the tree T and the bijection μ which give the minimum width, the *branchwidth*, denoted by $\mathbf{bw}(G)$. In Figure 2, one can find an example of a branch decomposition of a graph.

Checking whether the branchwidth of a graph is at most k is NP-complete when k is part of the input (see Seymour and Thomas [56]) while, if k is not part of the input, the problem can be solved in $O(n)$ steps due to Bodlaender and Thilikos [12, 57]. Also, the same problem is solvable in polynomial time when the input graph is planar [41, 56]. Finally, by Feige et al. [32], branchwidth is $O(1)$-approximable for any graph class excluding some fixed graph as a minor and admits an $OPT\sqrt{\log OPT}$-approximation for general graphs.

A *parameter* P is any function mapping graphs to nonnegative integers. The *parameterized problem associated with* P asks, for some fixed k, whether $P(G) = k$ for a given graph G. We say that a parameter P is *closed under taking of minors (contractions)* (or, briefly, *minor (contraction) closed*) if for every graph H, $H \preceq G$ ($H \preceq_c G$) implies that $P(H) \leq P(G)$.

The following three sample problems capture the most important properties of the investigated parameterized problems.

k-Vertex Cover. A *vertex cover* C of a graph is a set of vertices such that every edge of G has at least one endpoint in C (for example, the minimum size of a vertex cover in the graph G_1 in Figure 1 is five). The k-Vertex Cover problem is to decide, given a graph G and a positive integer k, whether G has a vertex cover of size k. Let us note that vertex cover is closed under taking minors, i.e. if a graph G has a vertex cover of size k, then each of its minors has a vertex cover of size at most k.

k-Dominating set. A *dominating* set D of a graph G is a set of vertices such that every vertex outside D is adjacent to a vertex of D (for example, the minimum size of a dominating set in the graph G_1 in Figure 1 is three). The k-Dominating Set problem is to decide, given a graph G and a positive integer k, whether G has a dominating set of size k. Let us note that the dominating set is not closed under taking minors (for example, graph G_1 in Figure 1 contains a matching of five edges that induce a graph where any dominating set has size at least 5). However, it is closed under contraction of edges.

Given a branch decomposition of G of width $\leq \ell$ both problems k-Vertex Cover and k-Dominating Set can be solved in time $2^{O(\ell)}n^{O(1)}$ (see $[1, 6, 8, 27, 37]$).

k-Longest path. The k-Longest Path problem is to decide, given a graph G and a positive integer k, whether G contains a path of length k (for example the longest path in the graph G_1 in Figure 1 is of length eight). The complement of this problem (asking whether G does not contain a path of length k) is closed under taking minors. The best known algorithm solving this problem on a graph of branchwidth $\leq \ell$ runs in time $2^{O(\ell \log \ell)}n^{O(1)}$ (this algorithm is based on standard dynamic programming techniques for graphs of bounded treewidth or branchwidth, see e.g. Bodlaender [9]).

The main idea behind the majority of subexponential parameterized graph algorithms $[1, 26, 36, 37, 42, 44]$ on graph classes $\mathcal{G}$ is that a parameter P satisfies the following two conditions for some constants α and β:

(A) For every graph $G \in \mathcal{G}$, $\mathbf{bw}(G) \leq \alpha \cdot \sqrt{P(G)} + O(1)$

(B) For every graph $G \in \mathcal{G}$ and given a branch decomposition (T, μ) of G, the value of $P(G)$ can be computed in $2^{\beta \cdot \mathbf{w}(T,\mu)}n^{O(1)}$ steps.

Conditions (A) and (B) are essential for Bidimensionality Theory due to the following generic result.

Theorem 1. *Let P be a parameter and let $\mathcal{G}$ be a class of graphs such that (A) and (B) hold for some constants α and β. Then, given a branch decomposition (T, μ) where $\mathbf{w}(T, \mu) \leq \lambda \cdot \mathbf{bw}(G)$ for a constant λ, the parameterized problem associated with P can be solved in $2^{O(\sqrt{k})}n^{O(1)}$ steps.*

Proof. Given a branch decomposition (T, μ) as above, one can solve the parameterized problem associated with P as follows. If $\mathbf{w}(T, \mu) > \lambda \cdot \alpha \cdot \sqrt{k}$, then the answer to the associated parameterized problem with parameter k is "NO" if it is a minimization and "YES" if it is a maximization problem. Else, by (B), $P(G)$ can be computed in $2^{\lambda \cdot \alpha \cdot \beta \cdot \sqrt{k}} n^{O(1)}$ steps. $\square$

To apply Theorem 1, we need an algorithm that computes, in polynomial time $t(n)$, a branch decomposition (T, μ) of any n-vertex graph $G \in \mathcal{G}$ such that $\mathbf{w}(T, \mu) \leq \lambda \cdot \mathbf{bw}(G) + O(1)$. This is possible for planar graphs due to the results of Seymour and Thomas in [56] where an algorithm is provided for computing an optimal branch decomposition of any planar graph in $O(n^4)$ steps (this algorithm has been improved later to an $O(n^3)$ step algorithm by Gu and Tamaki [41]). We stress that the algorithm in [56] is not involved, however the proof of its correctness is based on a min-max characterization of branchwidth emerging from the Graph Minors series of Robertson and Seymour (see [49, 50]). We conclude that $t(n) = n^{O(1)}$ and $\lambda = 1$ for planar graphs. For H-minor-free graphs (and thus, for all graph classes considered here), $t(n) = f(|H|) \cdot n^{O(1)}$ and $\lambda \leq f(|H|)$ for some function f depending only on the size of H (see [25, 29, 32]).

In this survey we discuss how

- to obtain a general scheme of proving bounds required by (A) and to extend parameterized algorithms to more general classes of graphs like graphs of bounded genus and graphs excluding a minor (Section 3);
- to improve the running times of such algorithms (Section 4), and
- to prove that the running time of many dynamic programming algorithms on planar graphs (and more general classes as well) satisfies (B) (Section 5).

3 Property (A) and bidimensionality

In this section we show how to obtain subexponential parameterized algorithms in the case when condition (B) holds for general graphs. The main tool for this is Bidimensionality Theory developed in [18, 20–22, 24]. For a survey on Bidimensionality Theory see Demaine and Hajiaghayi [17].

Planar graphs. While the results of this subsection can be extended to wider graph classes, we start from planar graphs, where the general ideas are easier to explain. The following theorem is the main ingredient for proving condition (A).

Theorem 2 ([55]). *Let $\ell \geq 1$ be an integer. Every planar graph of branchwidth $\geq \ell$ contains an $(\lceil \ell/4 \rceil \times \lceil \ell/4 \rceil)$-grid as a minor.*

The proof of Theorem 2 is based on the min-max theorem for branchwidth from Robertson and Seymour [49]. A simpler algorithmic proof of the same result (but with slightly worse constants when translated for branchwidth) can be found in Grigoriev [40]). Let us demonstrate the usefulness of Theorem 2 with the following examples.

We start with PLANAR k-VERTEX COVER. Let G be a planar graph of branchwidth $\geq \ell$. Observe that given a $(r \times r)$-grid H, the size of a vertex cover in H is at least $\lfloor r/2 \rfloor \cdot r$ (because of the existence of a matching of size $\lfloor r/2 \rfloor \cdot r$ in H). By Theorem 2, we have that G contains an $(\lceil \ell/4 \rceil \times \lceil \ell/4 \rceil)$-grid as a minor. The size of any vertex cover of this grid is at least $\ell^2/32$. As such a grid is a minor of G, and vertex cover is closed under minors, we conclude that G has a vertex cover of size at least $\ell^2/32$. This implies that if a planar graph has a vertex cover of size $\leq k$, then its branchwidth is upper bounded by $\sqrt{32} \cdot \sqrt{k}$. Therefore, property (A) holds for $\alpha = 4\sqrt{2}$.

For the PLANAR k-DOMINATING SET problem, the arguments used above to prove (A) for PLANAR k-VERTEX COVER do not work. Since the problem is not minor-closed, we cannot use Theorem 2 as above. However, since the parameter is closed under edge contractions, we can use a *partially triangulated $(r \times r)$-grid* which is any planar graph obtained from the $(r \times r)$-grid by adding some edges (see Figure 3). For every partially triangulated $(r \times r)$-grid H, the size of a dominating set in H is at least $\frac{(r-2)^2}{9}$ (every "inner" vertex of H has a closed neighborhood of at most 9 vertices). Theorem 2 implies that a planar graph G of branchwidth $\geq \ell$ can be contracted to a partially triangulated $(\lceil \ell/4 \rceil \times \lceil \ell/4 \rceil)$-grid which yields that PLANAR k-DOMINATING SET also satisfies (A) for $\alpha = 12$.

These two examples induce the following idea: if the graph parameter is closed under taking minors or contractions, the only thing needed for the proof of (A) is to understand how this parameter behaves on a (partially triangulated) grid. This brings us to the following definition.

Definition 3 ([20]). *A parameter P is* minor bidimensional with density δ *if*

1. *P is closed under taking of minors, and*

2. *for the $(r \times r)$-grid R, $P(R) = (\delta r)^2 + o((\delta r)^2)$.*

A parameter P is called contraction bidimensional with density δ *if*

1. *P is closed under contractions,*

2. *for any partially triangulated $(r \times r)$-grid R, $P(R) = (\delta_R r)^2 + o((\delta_R r)^2)$, and*

3. *δ is the smallest δ_R among all paritally triangulated $(r \times r)$-grids.*

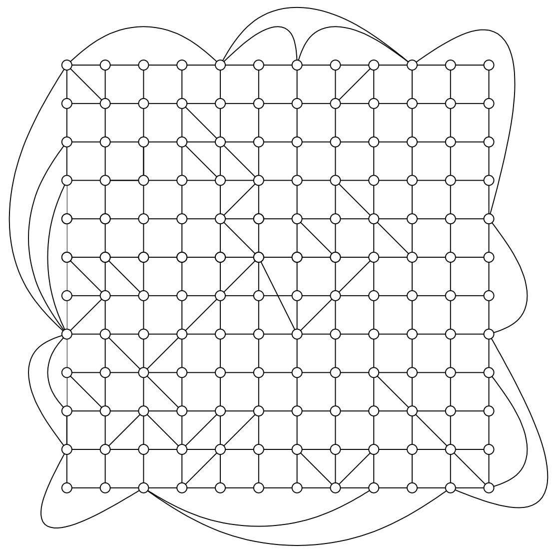

Figure 3: A partial triangulation of a (12×12)-grid.

In either case, P is called bidimensional. *The* density *δ of P is the minimum of the two possible densities (when both definitions are applicable), $0 < \delta \leq 1$.*

Intuitively, a parameter is bidimensional if its value depends on the "area" of a grid and not on its "height".

Many parameters are bidimensional. Some of them, like the number of vertices or the number of edges, are not so much interesting from the algorithmic point of view. Of course the already mentioned parameter *vertex cover* (*dominating set*) is minor (contraction) bidimensional (with densities $1/\sqrt{2}$ for *vertex cover* and $1/9$ for *dominating set*). Other examples of bidimensional parameters are *feedback vertex set* with density $\delta \in [1/2, 1/\sqrt{2}]$, *minimum maximal matching* with density $\delta \in [1/\sqrt{8}, 1/\sqrt{2}]$ and *longest path* with density 1.

By Theorem 2, we have the following.

Lemma 4. *If P is a bidimensional parameter with density δ then P satisfies property (A) for $\alpha = 4/\delta$, on planar graphs.*

By Lemma 4, Theorem 1 holds for every bidimensional parameter satisfying (B). Also, Theorem 1 can be applied not only to bidimensional parameters but to parameters that are bounded by bidimensional parameters. For example, the *clique-transversal number* of a graph G is the minimum number of vertices intersecting every maximal clique of G. This parameter is not contraction-closed because an edge contraction may create a new maximal clique and cause the clique-transversal number to increase (for example, in Figure 1, the clique transversal number of graph G_1 is three, while the clique transversal number of the graph G_2 is four). On the other hand, it is easy to see that this graph parameter always exceeds the size of any minimum dominating set which yields (A) for this parameter.

Non-planar extensions and limitations. One of the natural approaches of extending Lemma 4 from planar graphs to more general classes of graphs is via a generalization of Theorem 2. To do this we have to treat separately minor closed and contraction closed parameters.

For graphs embedded in surfaces, it is convenient to work with Euler genus. The *Euler genus* of a nonorientable surface Σ is equal to the nonorientable genus $\tilde{g}(\Sigma)$ (or the crosscap number). The *Euler genus* of an orientable surface Σ is $2g(\Sigma)$, where $g(\Sigma)$ is the orientable genus of Σ. We refer to the book of Mohar and Thomassen [46] for information on graphs embeddings.

The following extension of Theorem 2 holds for bounded genus graphs:

Theorem 5 ([20])**.** *If G is a graph of Euler genus at most γ with branch-width more than r, then G contains a $(\lceil r/4(\gamma+1)\rceil \times \lceil r/4(\gamma+1)\rceil)$-grid as a minor.*

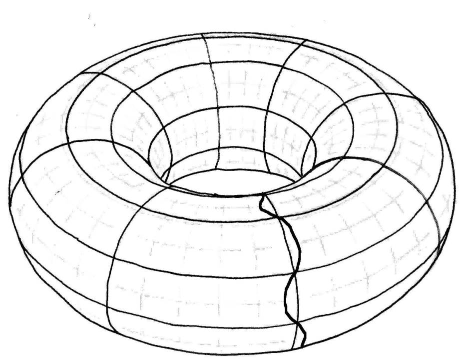

Figure 4: A orientable surface of genus one (torus) and a toroidal grid embedded in it. The bold line is the visible part of a non-contractible noose.

The proof of Theorem 5, is strongly based on the notion of representativity of a graph embedding defined by Robertson and Seymour in [51] and studied in [52]. Let G be a planar graph embedded in a surface $\mathbb{S}$. An O-arc is a subset of $\mathbb{S}$ homeomorphic to a circle. An O-arc in $\mathbb{S}$ is called a *noose* of the embedding of G if it meets G only in vertices. The length of a noose O is the number of vertices of G it meets. The *representativity* of a graph embedded in a surface is the minimum number of vertices in a non-contractible noose. (Some authors also use the name face-width for this parameter.) The book of Mohar and Thomassen [46] contains a chapter devoted to this parameter. According to Demaine et al. [20], any embedding of representativity at least $4r$ contains as a minor an $(r \times r)$-grid. If the representativity is at most $4r$, then there is a noose meeting few vertices along which we can "split" the graph to one that is embeddible to a surface of lower genus where the same arguments are repeated recursively.

Working analogously to the planar case, Theorem 5 implies the following.

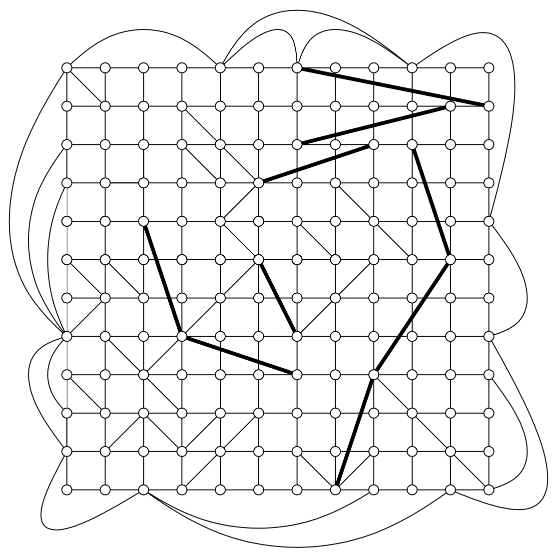

Figure 5: A $(12, 9)$-gridoid.

Lemma 6. *Let P be a minor bidimensional parameter with density δ. Then for any graph G of Euler genus at most γ, property (A) holds for $\alpha = 4(\gamma + 1)/\delta$.*

The next step is to consider graphs excluding a fixed graph H as a minor. The proof extends Theorem 5 by making (nontrivial) use of the structural characterization of H-minor-free graphs by Robertson and Seymour in [54]. (We discuss this characterization in Section 5.)

Theorem 7 ([21])**.** *If G is an H-minor-free graph with branchwidth more than r, then G has the $(\Omega(r) \times \Omega(r))$-grid as a minor (the hidden constants in the Ω notation depend only on the size of H).*

As before, Theorem 5 implies property (A) for all minor bidimensional parameters for some α depending only on the excluded minor H.

For contraction-closed parameters, the landscape is different. In fact, each possible extension of Lemma 6, requires a stronger version of bidimensionality. For this, we can use the notion of a (r, q)-*gridoid* that is obtained from a partially triangulated $(r \times r)$-grid by adding at most q edges, see Figure 5. (Note that every (r,q)-gridoid has genus $\leq q$.) The following extends Theorem 2 for graphs of bounded genus.

Theorem 8 ([20])**.** *If a graph G of Euler genus at most γ excludes all $(k-12\gamma,\gamma)$-gridoids as contractions, for some $k \geq 12\gamma$, then G has branchwidth at most $4k(\gamma+1)$.*

A parameter is *genus-contraction bidimensional* if *a*) it is contraction closed and *b*) its value on every $(r, O(1))$-gridoid is $\Omega(r^2)$ (here the hidden constants in the "O" and the "Ω" notations depend only on the Euler genus). Then Theorem 8 implies property (A) for all genus-contraction bidimensional parameters for some constant that depends only on the Euler genus.

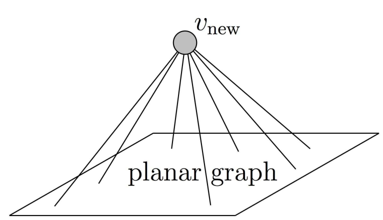

Figure 6: An apex graph.

An *apex graph* is a graph obtained from a planar graph G by adding a new vertex v_{new} and making it adjacent to some vertices of G (see Figure 6).

A graph class is *apex-minor-free* if it does not contain a graph with some fixed apex graph as a minor. An (r,s)-*augmented grid* is an $(r \times r)$-grid with some additional edges such that each vertex is attached to at most s vertices that in the original grid had degree 4. (An example of a $(12,8)$-augmented grid is given in Figure 7. The black vertex has 8 neighbors that have degree 4 in the underlying (12×12)-grid.) We say that a contraction closed parameter P is *apex-contraction bidimensional* if *a*) it is closed under taking of contractions and *b*) its value on every $(r, O(1))$-augmented grid is $\Omega(r^2)$ (here the hidden constants in the "O" and the "Ω" notations depend only on the excluded apex graph). As it was shown by Demaine et al. [18], every apex-minor free graph with treewidth at least k can be contracted to a $(f(k)), O(1))$-augmented. Because, $f(k) = \Omega(k)$ (due to the results of Demaine and Hajiaghayi [21]), every apex-contraction bidimensional parameter satisfies property (A) for some constant that depends only on the excluded apex graph.

A natural question appears: until what point property (A) can be satisfied for contraction-closed parameters (assuming a suitable concept of

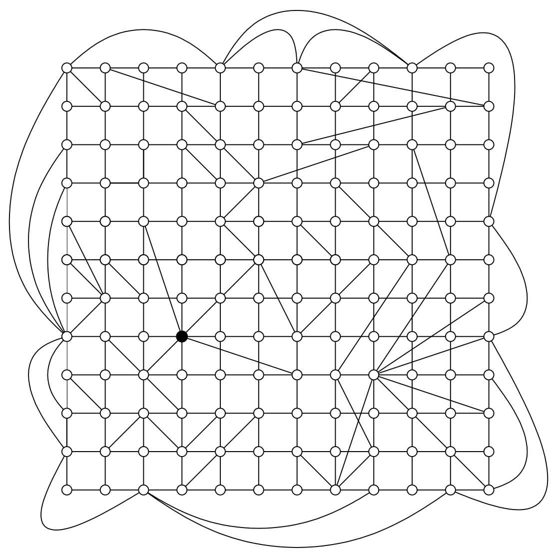

Figure 7: A $(12, 8)$-augmented grid.

bidimensionality)? As it was observed by Demaine et al. [18], for some contraction-closed parameters, like dominating set, the branchwidth of an apex graph cannot be bounded by any function of their value: just take the graph G obtained by an $(n \times n)$-grid after connecting all its vertices with a new vertex. Its branchwidth is $\Omega(n)$, while the new vertex dominates all other vertices in G. Consequently, apex-free graph classes draw a natural combinatorial limit on the the above framework of obtaining subexponential parameterized algorithms for contraction-closed parameters. (On the other side, this is not the case for minor-closed parameters as it is indicated by Theorem 7, see Figure 8.) However, it is still possible to cross the frontier of apex-minor-free graphs for the dominating set problem and some of its variants where subexponential parameterized algorithms exist, even for H-minor-free graphs, as it is shown in [20]. These algorithms are based on a combination of dynamic programming and the structural characterization of H-minor-free graphs from Robertson and Seymour [54]. For recent and more general results in this direction, see Demaine et al. [23].

4 Further optimizations

In this section, we present several techniques for accelerating the algorithms emerging by the framework of Theorem 1.

Making algorithms faster. While proving properties (A) and (B), it is natural to ask for the best possible constants α and β, as this directly implies an exponential speed-up of the corresponding algorithms. While,

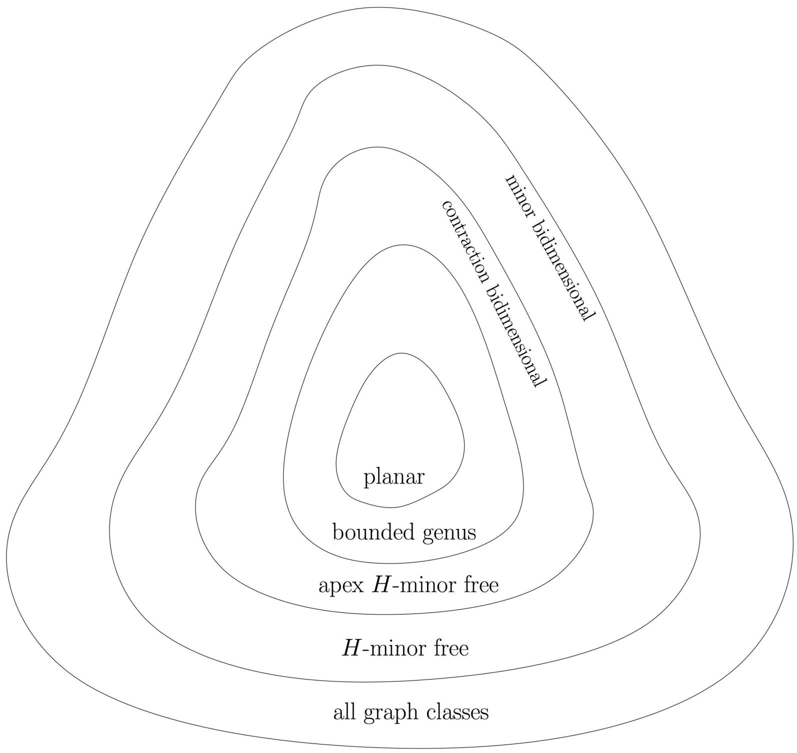

Figure 8: The territories of the applicability of Bidimensionality Theory.

Bidimensionality Theory provides some general estimation of α, in some cases, deep understanding of the parameter behavior can lead to much better constants in (A). For example, it holds that for Planar k-Vertex Cover, $\alpha \leq 3$ (see Fomin and Thilikos [38]) and for Planar k-Dominating Set, $\alpha \leq 6.364$ (see Fomin and Thilikos [37]). (Both bounds are based on the fact that planar graphs with n vertices have branchwidth at most $\sqrt{4.5}\sqrt{n}$, see [38].) Similar results hold also for bounded genus graphs [36].

On the other hand, there are several ways to obtain faster dynamic programming algorithms and to obtain better bounds for β in (B). Adapting the treewidth techniques (see Arnborg and Proskurowski [6], and Bodlaender [11]) on branch decompositions, a typical approach computing an optimal solution to a problem is as follows:

- In a given branch decomposition (T, μ) of a graph G we select a root r by picking (arbitrarily) one of the vertices of T and by applying dynamic programming on the middle sets, starting from the leaves and moving towards the root.

- Each middle set $\mathbf{mid}(e)$ of (T, μ) represents the subgraph G_e of G formed by the edges of G which correspond to the leaves of T below e.

- In each step of the dynamic programming, all optimal solutions for a subproblem in G_e are computed, subject to all possibilities of how $\mathbf{mid}(e)$ contributes to an overall solution for G. E.g., for Vertex Cover, there are up to $2^{\mathbf{w}(T,\mu)}$ subsets of $\mathbf{mid}(e)$ that may constitute a vertex cover of G.

- The partial solutions of a middle set are computed using those of the already processed middle sets of the children and stored by making use of an appropriate data structure.

- An optimal solution to the problem is computed at the root of T.

Encoding the middle sets in a refined way may speed up the processing time significantly. There are some methods to accelerate the update of the solutions of two middle sets to a parent middle set:

Using the right data structure: storing the solutions in a sorted list reduces the time consuming search for compatible solutions and allows a fast computing of the new solution. E.g., for k-Vertex Cover, the time to process two middle sets is reduced from $O(2^{3\cdot\mathbf{w}(T,\mu)})$ (for each subset of the parent middle set, all pairs of solutions of the two children are computed) to $O(2^{1.5\cdot\mathbf{w}(T,\mu)})$. In Dorn [27] matrices are used as a data structure for dynamic programming that allows an updating even in time $O(2^{\frac{\omega}{2}\mathbf{w}(T,\mu)})$ for k-Vertex Cover (where ω is the fast matrix multiplication constant, actually $\omega < 2.376$). Even though the currently best constant $\omega < 2.376$ of fast matrix multiplication is of rather theoretical interest, there exist some practical sub-cubic runtime algorithms that help improving the runtime for solving all mentioned problems.

A compact encoding: assign as few as possible vertex states to the vertices and reduce the number of processed solutions. Alber et al. [1], using the so-called "monotonicity technique", showed that 3 vertex states are sufficient in order to encode a solution of k-Dominating Set. A similar approach was used by Fomin and Thilikos [37] to obtain, for the same problem, an $O(3^{1.5\cdot\mathbf{w}(T,\mu)})$-step updating process, that has been improved by Dorn [27] to $O(2^{2\cdot\mathbf{w}(T,\mu)})$.

Subset convolution: Björklund et al. [7] introduced a fast algorithm for the subset convolution problem: given functions f and g defined on the lattice of subsets of an n-element set N, compute their subset convolution $f \star g$, defined for all $S \subseteq N$ by $(f \star g)(S) = \sum_{T\subseteq S} f(T)g(S \setminus T)$, where addition and multiplication is carried out in an arbitrary ring. Rossmanith, in [16], observed how this technique can be used to speed up dynamic programming on graphs of bounded treewidth.

Exploiting graph structures: as we will see in Section 5, one can improve the runtime further for dynamic programming on branch decompositions whose middle sets inherit some structure of the graph. By exploiting the planarity of the input graph, the update process for PLANAR k-DOMINATING SET can be done in time $O(3^{\frac{\omega}{2}\mathbf{w}(T,\mu)})$ [27].

The above techniques can be used to prove the following result.

Theorem 9 ([27]). PLANAR k-VERTEX COVER *can be solved in* $O(2^{3.56\sqrt{k}}) \cdot n^{O(1)}$ *runtime and* PLANAR k-DOMINATING SET *in* $O(2^{11.98\sqrt{k}}) \cdot n^{O(1)}$ *runtime.*

Kernels. Many of the parameterized algorithms discussed in this section can be further accelerated to time $O(n^{\theta}) + 2^{O(\sqrt{k})}$ for θ being a small integer (usually ranging from 1 to 3). This can be done using the technique of kernelization that is a prolynomial step preprocessing of the initial input of the problem towards creating an equivalent one, whose size depends exclusively on the parameter. Examples of such problems are PLANAR k-DOMINATING SET [4, 14, 36], k-FEEDBACK VERTEX SET [10], k-VERTEX COVER and others [33]. As kernel constructions are out of the scope of this survey, we address the reader to the books [31, 35, 47] for further references.

5 Property (B) and Catalan structures

All results of the previous sections provide subexponential parameterized algorithms when property (B) holds. However, there are many bidimensional parameters for which there is no known algorithm providing property (B) in general. The typical running times of dynamic programming algorithms for these problems are $O(\mathbf{bw}(G)!) \cdot n^{O(1)}$, $O(\mathbf{bw}(G)^{\mathbf{bw}(G)}) \cdot n^{O(1)}$, or even $O(2^{\mathbf{bw}(G)^2}) \cdot n^{O(1)}$. Examples of such problems are parameterized versions of k-LONGEST PATH, k-FEEDBACK VERTEX SET, k-CONNECTED DOMINATING SET, and k-GRAPH TSP (a version of metric TSP with metric being the shortest path metric of some graph). Usually, these problems are in NP whose certificate verification involves some connectivity question. In this section, we show that for such problems one can prove that (B) actually holds for the graph class that we are interested in. To do this, one has to make further use of the structure of the input graphs (again using ideas from Graph Minors Theory) that can vary from planar graphs to H-minor-free graphs. In other words, we use the structure of the graph class not only for proving (A) but also for proving (B).

Planar graphs. The following type of decomposition for planar graphs follows from the results of Seymour and Thomas (Theorem (5.1) in [56]) and is extremely useful for making dynamic programming on graphs of bounded branchwidth faster (see [27, 30]).

Let G be a planar graph embedded in a sphere $\mathbb{S}$. Every noose O in $\mathbb{S}$. bounds two open discs Δ_1, Δ_2 in $\mathbb{S}$, i.e., $\Delta_1 \cap \Delta_2 = \emptyset$ and $\Delta_1 \cup \Delta_2 \cup O = \mathbb{S}$.

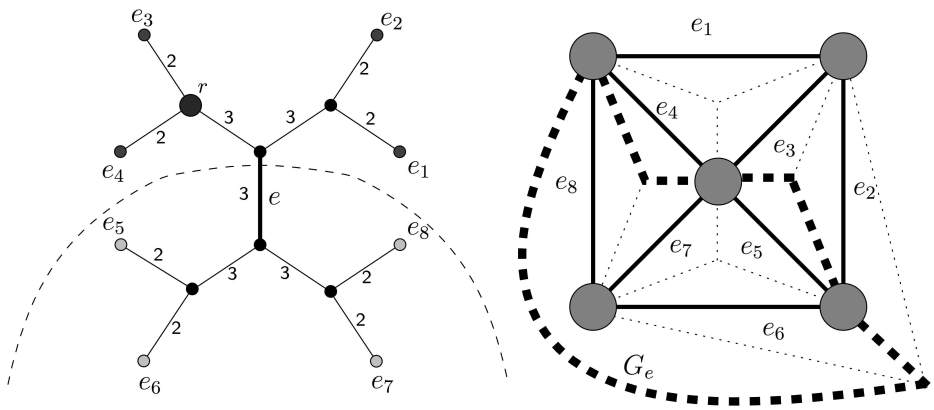

Figure 9: A sphere cut decomposition and the noose corresponding to edge e.

We define a *sphere cut decomposition* or *sc-decomposition* (T, μ, π) as a branch decomposition with the following property: for every edge e of T, there exists a noose O_e meeting every face at most once and bounding the two open discs Δ_1 and Δ_2 such that $G_i \subseteq \Delta_i \cup O_e$, $1 \leq i \leq 2$. Figure 9 shows an example of a sphere cut decomposition. Thus O_e meets G only in $\mathbf{mid}(e)$ and its length is $|\mathbf{mid}(e)|$. A *clockwise traversal* of O_e in the embedding of G defines the cyclic ordering π of $\mathbf{mid}(e)$. We always assume that the vertices of every middle set $\mathbf{mid}(e) = V(G_1) \cap V(G_2)$ are enumerated according to π. The following theorem follows from the results of Seymour and Thomas [56] (see also Dorn et al. [30]).

Theorem 10. *Let G be a planar graph of branchwidth at most ℓ without vertices of degree one embedded on a sphere. Then there exists an sc-decomposition of G of width at most ℓ that can be constructed in time $O(n^3)$.*

Theorem 10 indicates that it is possible to consider an optimal branch decomposition of a plane graph where each middle set is situated cyclically on the plane where G is embedded.

In what follows, we sketch the main idea of a $2^{O(\mathbf{w}(T,\mu,\pi))} n^{O(1)}$ algorithm for the k-Planar Longest Path. One may use k-Longest path as an example for other problems of the same nature.

The algorithm follows the dynamic programming scheme described in Section 4. A state of the dynamic programming algorithm associated to an edge e of the tree T of the sc-decomposition (T, μ, π), is a set of non-crossing pairs of vertices in $\mathbf{mid}(e)$ that, in turn, correspond to non-crossisng pairs of paths in the embedded graph G_e induced by the edges "not below" edge e in (T, μ, π). As the vertices of $\mathbf{mid}(e)$ are cyclically arranged on a noose of the plane, the number of non-crossing partitions is bounded by the Catalan number of $\mathbf{mid}(e)$, that is singly exponentially on the branchwidth of G. Based on this idea, it is possible to reduce drastically the number of states in dynamic programming.

Formally, to count the number of states at each step of the dynamic programming, we should estimate the number of collections of internally vertex disjoint paths using edges from $E \subseteq E(G)$ and having their (different) endpoints in $S \subseteq V(G)$. We use the notation $\mathbf{P}$ to denote such a path collection and we define $\mathbf{paths}_G(E, S)$ as the set of all such path collections. Define an equivalence relation $\sim$ on $\mathbf{paths}_G(E, S)$: for $\mathbf{P}_1, \mathbf{P}_2 \in \mathbf{paths}_G(E, S)$, $\mathbf{P}_1 \sim \mathbf{P}_2$ if there is a bijection between $\mathbf{P}_1$ and $\mathbf{P}_2$ such that bijected paths in $\mathbf{P}_1$ and $\mathbf{P}_2$ have the same endpoints. Denote by $\texttt{q-paths}_G(E, S) = |\mathbf{paths}_G(E, S)/\sim|$ the cardinality of the quotient set of $\mathbf{paths}_G(E, S)$ by $\sim$.

Recall that we define $\texttt{q-paths}_G(E, S)$ because, while applying dynamic programming on some middle set $\mathbf{mid}(e)$ of the branch decomposition (T, μ), the number of states for $e \in E(T)$ is bounded by $O(\texttt{q-paths}_{G_i}(E(G_i), \mathbf{mid}(e)))$.

Given a graph G and a branch decomposition (T, μ) of G, we say that (T, μ) has *Catalan structure* if for every edge $e \in E(T)$ and any $i \in \{1, 2\}$,

$$\texttt{q-paths}_{G_i}(E(G_i), \mathbf{mid}(e)) = 2^{O(\mathbf{w}(T,\mu))} \tag{1}$$

Now, (B) holds for planar graphs because of the following combinatorial result.

Theorem 11 ([30]). *Every planar graph has an optimal branch decomposition with the Catalan structure that can be constructed in polynomial time.*

The proof of Theorem 11 uses a sc-decomposition (T, μ, π) (constructed by using the polynomial algorithm of Seymour and Thomas [56]). Let O_e be a noose meeting some middle set $\mathbf{mid}(e)$ of (T, μ, π). Let us count in how many ways this noose can cut paths of G. Observe that each path is cut into at most $\mathbf{w}(T, \mu, \pi)$ parts. Each such part is itself a path whose endpoints are pairs of vertices in O_e. Notice also that, because of planarity, no two such pairs can cross. Therefore, counting the ways O_e can intersect paths of G is equivalent to counting non-crossing pairs of vertices in a cycle (the noose) of length $\mathbf{w}(T, \mu, \pi)$ which, in turn, is bounded by the Catalan number of $\mathbf{w}(T, \mu, \pi)$ that is $2^{O(\mathbf{w}(T,\mu,\pi))}$.

We just concluded that the application of dynamic programming on an sc-decomposition (T, μ, π) is the $2^{O(\mathbf{w}(T,\mu,\pi))} n^{O(1)}$ algorithm for proving property (B) for planar graphs. By further improving the way the members of $\texttt{q-paths}_{G_i}(E(G_i), \mathbf{mid}(e))$ are encoded during this procedure, one can bound the hidden constants in the "O" notation on the exponent of this algorithm (see Dorn et al. [30]). For example, for Planar k-Longest Path $\beta \leq 2.63$. With analogous structures and arguments it follows that for Planar k-Graph TSP $\beta \leq 3.84$, for Planar k-Connected Dominating Set $\beta \leq 3.82$, for Planar k-Feedback Vertex Set $\beta \leq 3.56$ (see Dorn [27]).

In Dorn et al. [28], all above results were generalized for graphs with bounded genus (now constants for each problem depend also on the genus). This generalization requires a suitable "bounded genus" extension of Theorem 11 and its analogues for other problems.

Excluding a minor. The final step is to prove property (B) for H-minor-free graphs. For the proof of this, we need the following analogue of Theorem 11.

Theorem 12 ([29])**.** *Let $\mathcal{G}$ be a graph class excluding some fixed graph H as a minor. Then every graph $G \in \mathcal{G}$ with $\mathbf{bw}(G) \leq \ell$ has an branch decomposition of width $O(\ell)$ with the Catalan structure (here the hidden constants in the "O" notations in $O(\ell)$ and the upper bound certifying the Catalan structure in Equation* (1) *depend only on H). Moreover, such a decomposition can be constructed in $f(|H|)\cdot n^{O(1)}$ steps, where f is a function depending only on H.*

The proof of Theorem 12 is based on an algorithm constructing the claimed branch decomposition using the structural characterization of H-minor-free graphs of Robertson and Seymour [54]. Briefly, any H-minor-free graph can be seen as the result of gluing together (identifying constant size cliques and, possibly, removing some of their edges) graphs that, after the removal of some constant number of vertices (called *apices*) can be "almost" embedded in a surface of constant genus. Here, by "almost" we mean that we permit a constant number of non-embedded parts (called *vortices*) that are "attached" around empty disks of the embedded part and have a path-like structure of constant width. The algorithm of Theorem 12, as well as the proof of its correctness, has several phases, each dealing with some level of this characterisation, where an analogue of sc-decomposition for planar graphs is used. The core of the proof is based on the fact that the structure of the embeddible parts of this characterisation (along with vortices) is "close enough" to be planar, so to roughly maintain the Catalan structure property.

Theorem 12 implies (B) for k-Longest Path on H-minor-free graphs. Similar results can be obtained for all problems examined in this section on H-minor-free graphs. Since property (A) holds for minor/apex-contraction bidimensional parameters on H-minor-free/apex-minor-free graphs, we have that one can design parameterized algorithms with running time $2^{O(\sqrt{k})}\cdot n^{O(1)}$ for all problems examined in this section for H-minor-free/ apex-minor-free graphs (here the hidden constant in the "O" notation in the exponent depends on the size on the excluded minor).

6 Conclusion

In Section 3, we have seen that bidimensionality can serve as a general combinatorial criterion implying property (A). Moreover, no such a characterization is known, so far, for proving property (B). In Section 5, we have presented several problems where an analogue of Theorem 12 can be proven, indicating the existence of Catalan structures in H-minor-free graphs. It would be challenging to find a classification criterion (logical or combinatorial) for the problems that are amenable to this approach. Another interesting direction of (related) research would be a development of complexity theory for obtaining lower bounds on the running time of dynamic programming algorithms on graphs of bounded treewidth. For example, for general graphs of treewidth ℓ, how fast can we find a longest path? Is it $O(2^{o(\ell \log \ell)} \cdot n^c)$ (where c is some universal constant), or $O(2^{O(\ell \log \ell)} \cdot n^c)$ is the best we can hope for (up to some assumption in Complexity Theory)? Or, is it possible to prove that the Maximum Independent Set problem in a graph on n vertices and of treewidth ℓ cannot be solved, say, in time $O(1.1^{\ell} \cdot n^c)$?

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References

[1] J. Alber, H. L. Bodlaender, H. Fernau, T. Kloks, and R. Niedermeier, *Fixed parameter algorithms for dominating set and related problems on planar graphs*, Algorithmica, 33 (2002), pp. 461–493.

[2] J. Alber, H. L. Bodlaender, H. Fernau, and R. Niedermeier, *Fixed parameter algorithms for planar dominating set and related problems*, Proceedings of 7th Scandinavian Workshop on Algorithm Theory (SWAT 2000), vol. 1851 of LNCS, Springer, 2000, pp. 97–110.

[3] J. Alber, H. Fan, M. R. Fellows, H. Fernau, R. Niedermeier, F. A. Rosamond, and U. Stege, *Refined search tree technique for dominating set on planar graphs.*, Proceedings of the 26th International Symposium on Mathematical Foundations of Computer Science (MFCS 2001), vol. 2136 of LNCS, Springer, 2001, pp. 111–122.

[4] J. Alber, M. R. Fellows, and R. Niedermeier, *Polynomial-time data reduction for dominating set*, Journal of the ACM, 51 (2004), pp. 363–384.

[5] J. Alber, H. Fernau, and R. Niedermeier, *Parameterized complexity: exponential speed-up for planar graph problems*, J. Algorithms, 52 (2004), pp. 26–56.

[6] S. Arnborg and A. Proskurowski, *Linear time algorithms for NP-hard problems restricted to partial k-trees*, Discr. Appl. Math., 23 (1989), pp. 11–24.

[7] A. Björklund, T. Husfeldt, P. Kaski, and M. Koivisto, *Fourier meets Möbious: Fast subset convolution*, Proceedings of the 39th annual ACM Symposium on Theory of Computing (STOC 2007), New York, 2007, ACM Press, pp. 67-74.

[8] H. Bodlaender, *A Tourist Guide through Treewidth*, Acta Cybernetica, 11 (1993), pp. 1–23.

[9] ——, *On Linear Time Minor Tests with Depth-First Search*, J. Algorithms, 14 (1993), pp. 1-23.

[10] ——, *A cubic kernel for feedback vertex set*, Proceedings of the 24th Annual Symposium on Theoretical Aspects of Computer Science Symposium on Theoretical Aspects of Computer Science (STACS 2007), vol. 4393 of LNCS, Springer, 2007, pp. 320–331.

[11] ——, *Treewidth: Algorithmic techniques and results*, Proceedings of the 22nd International Symposiumin Mathematical Foundations of Computer Science (MFCS 1997) vol. 1295 of LNCS, Springer, 1997, pp. 19–36.

[12] H. L. Bodlaender and D. M. Thilikos, *Constructive linear time algorithms for branchwidth*, Procedeengs of the 24th International Colloquium Automata, Languages and Programming (ICALP 1997), vol. 1256 of LNCS, Springer, Berlin, 1997, pp. 627–637.

[13] L. Cai and D. Juedes, *On the existence of subexponential parameterized algorithms*, J. Comput. System Sci., 67 (2003), pp. 789–807.

[14] J. Chen, H. Fernau, I. A. Kanj, and G. Xia, *Parametric duality and kernelization: Lower bounds and upper bounds on kernel size*, in Proceedings of the 22nd International Symposium on Theoretical Aspects of Computer Science (STACS), vol. 3403 of LNCS, Springer, 2005, pp. 269–280.

[15] V. G. Deĭneko, B. Klinz, and G. J. Woeginger, *Exact algorithms for the Hamiltonian cycle problem in planar graphs*, Operations Research Letters, 34(3) (2006), pp. 269-274.

[16] E. Demaine, G. Z. Gutin, D. Marx, and U. Stege, *07281 Abstracts Collection – Structure Theory and FPT Algorithmics for Graphs, Digraphs and Hypergraphs*, Dagstuhl Seminar Proceedings, Internationales Begegnungs- und Forschungszentrum für Informatik (IBFI), Schloss Dagstuhl, Germany, n. 07281, 2007, p. 11, http://drops.dagstuhl.de/opus/volltexte/2007/1245

[17] E. Demaine and M. Hajiaghayi, *The bidimensionality theory and its algorithmic applications*, The Computer Journal, (2007), pp. 332–337.

[18] E. D. Demaine, F. V. Fomin, M. Hajiaghayi, and D. M. Thilikos, *Bidimensional parameters and local treewidth*, SIAM J. Discrete Math., 18 (2004), pp. 501–511.

[19] ——, *Fixed-parameter algorithms for (k, r)-center in planar graphs and map graphs*, ACM Trans. Algorithms, 1 (2005), pp. 33–47.

[20] ——, *Subexponential parameterized algorithms on graphs of bounded genus and H-minor-free graphs*, Journal of the ACM, 52 (2005), pp. 866–893.

[21] E. D. Demaine and M. Hajiaghayi, *Linearity of grid minors in treewidth with applications through bidimensionality*, Combinatorica, to appear.

[22] ——, *Bidimensionality: new connections between fpt algorithms and ptass*, in Proceedings of the 16th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2005), New York, 2005, ACM-SIAM, pp. 590–601.

[23] E. D. Demaine, M. Hajiaghayi, and K. Kawarabayashi, *Decomposition, approximation, and coloring of odd-minor-free graphs*, manuscript.

[24] E. D. Demaine, M. Hajiaghayi, and D. M. Thilikos, *The bidimensional theory of bounded-genus graphs*, SIAM J. Discrete Math., 20 (2006), pp. 357–371 (electronic).

[25] E. D. Demaine, M. T. Hajiaghayi, N. Nishimura, P. Ragde, and D. M. Thilikos, *Approximation algorithms for classes of graphs excluding single-crossing graphs as minors*, J. Comput. System Sci., 69 (2004), pp. 166–195.

[26] E. D. Demaine, M. T. Hajiaghayi, and D. M. Thilikos, *Exponential speedup of fixed-parameter algorithms for classes of graphs excluding single-crossing graphs as minors*, Algorithmica, 41 (2005), pp. 245–267.

[27] F. Dorn, *Dynamic programming and fast matrix multiplication*, Proceedings of the 14th Annual European Symposium on Algorithms (ESA 2006), vol. 4168 of LNCS, Springer, 2006, pp. 280–291.

[28] F. Dorn, F. V. Fomin, and D. M. Thilikos, *Fast subexponential algorithm for non-local problems on graphs of bounded genus*, Proceedings of the 10th Scandinavian Workshop on Algorithm Theory (SWAT 2006), LNCS, Springer, 2006, pp. 172–183.

[29] ——, *Catalan structures and dynamic programming on H-minor-free graphs*, in Proceedings of the 19th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2008), ACM-SIAM, to appear.

[30] F. Dorn, E. Penninkx, H. Bodlaender, and F. V. Fomin, *Efficient exact algorithms on planar graphs: Exploiting sphere cut branch decompositions*, Proceedings of the 13th Annual European Symposium on Algorithms (ESA 2005), vol. 3669 of LNCS, Springer, 2005, pp. 95–106.

[31] R. G. Downey and M. R. Fellows, *Parameterized complexity*, Springer-Verlag, New York, 1999.

[32] U. Feige, M. T. Hajiaghayi, and J. R. Lee, *Improved approximation algorithms for minimum-weight vertex separators [extended abstract]*, in Proceedings of the 37th Annual ACM Symposium on Theory of Computing (STOC 2005), New York, 2005, ACM, pp. 563–572.

[33] M. R. Fellows, *Blow-ups, win/win's, and crown rules: Some new directions in fpt.*, Proceedings of the 29th International Workshopin Graph-Theoretic Concepts in Computer Science, (WG 2003), vol. 2880 of LNCS, Springer, 2003, pp. 1–12.

[34] H. Fernau and D. W. Juedes, *A geometric approach to parameterized algorithms for domination problems on planar graphs.*, Proceedings of the 29th International Symposium Mathematical Foundations of Computer Science (MFCS), vol. 3153 of LNCS, Springer, 2004, pp. 488–499.

[35] J. Flum and M. Grohe, *Parameterized Complexity Theory*, Springer-Verlag, Berlin, 2006.

[36] F. V. Fomin and D. M. Thilikos, *Fast parameterized algorithms for graphs on surfaces: Linear kernel and exponential speed-up*, Proceedings of the 31st International Colloquium on Automata, Languages

and Programming (ICALP 2004), vol. 3142 of LNCS, Springer, 2004, pp. 581–592.

[37] ——, *Dominating sets in planar graphs: Branch-width and exponential speed-up*, SIAM J. Comput., 36 (2006), pp. 281–309.

[38] ——, *New upper bounds on the decomposability of planar graphs*, Journal of Graph Theory, 51 (2006), pp. 53–81.

[39] M. R. Garey and D. S. Johnson, *Computers and Intractability, A Guide to the Theory of NP-Completeness*, W.H. Freeman and Company, New York, 1979.

[40] A. Grigoriev, *Large grid minors in planar graphs, manuscript*, 2007.

[41] Q.-P. Gu and H. Tamaki, *Optimal branch-decomposition of planar graphs in $O(n^3)$ time*, Proceedings of the 32nd International Colloquium on Automata, Languages and Programming (ICALP 2005), vol. 3580 of LNCS, Springer, Berlin, 2005, pp. 373–384.

[42] G. Gutin, T. Kloks, C. M. Lee, and A. Yeo, *Kernels in planar digraphs*, J. Comput. System Sci., 71 (2005), pp. 174–184.

[43] R. Impagliazzo, R. Paturi, and F. Zane, *Which problems have strongly exponential complexity*, J. Comput. System Sci., 63 (2001), pp. 512–530.

[44] I. Kanj and L. Perković, *Improved parameterized algorithms for planar dominating set*, in Proceedings of the 27th International Symposium Mathematical Foundations of Computer Science (MFCS 2002), vol. 2420 of LNCS, Springer, Berlin, 2002, pp. 399–410.

[45] A. Levin, D. Paulusma, and G. J. Woeginger, *The complexity of graph contractions*, Proceedings of the 29th International Workshop on Graph-theoretic Concepts in Computer Science (WG 2003), vol. 2880 of LNCS, Springer, Berlin, 2003, pp. 322–333.

[46] B. Mohar and C. Thomassen, *Graphs on surfaces*, Johns Hopkins University Press, Baltimore, MD, 2001.

[47] R. Niedermeier, *Invitation to fixed-parameter algorithms*, Oxford University Press, Oxford, 2006.

[48] N. Robertson and P. D. Seymour, *An outline of a disjoint paths algorithm*, Paths, Flows and VLSI Design, Algorithms and Combinatorics, 9 (1990), pp. 267–292.

[49] N. Robertson and P. D. Seymour, *Graph minors. X. Obstructions to tree-decomposition*, J. Combin. Theory Ser. B, 52 (1991), pp. 153–190.

[50] N. Robertson and P. D. Seymour, *Graph minors. XI. Circuits on a surface*, J. Combin. Theory Ser. B, 60 (1994), pp. 72–106.

[51] N. Robertson and P. D. Seymour, *Graph minors. XI. Circuits on a surface*, J. Combin. Theory Ser. B, 60 (1994), pp. 72–106.

[52] ——, *Graph minors. XII. Distance on a surface*, J. Combin. Theory Ser. B, 64 (1995), pp. 240–272.

[53] ——, *Graph minors. XIII. The disjoint paths problem*, Journal of Combinatorial Theory. Series B, 63 (1995), pp. 65–110.

[54] ——, *Graph minors. XVI. Excluding a non-planar graph*, J. Combin. Theory Ser. B, 89 (2003), pp. 43–76.

[55] N. Robertson, P. D. Seymour, and R. Thomas, *Quickly excluding a planar graph*, J. Combin. Theory Ser. B, 62 (1994), pp. 323–348.

[56] P. D. Seymour and R. Thomas, *Call routing and the ratcatcher*, Combinatorica, 14 (1994), pp. 217–241.

[57] D. M. Thilikos and H. L. Bodlaender, *Constructive linear time algorithms for branchwidth*, Tech. Rep. UU-CS-2000-38, Dept. of Computer Science, Utrecht University, 2000.